HOMEWORK 2

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1. Prove the Triangle Inequality for Integrals:

$$\left|\int_a^b h(t)dt\right| \leq \int_a^b |h(t)|dt.$$

Hint: Write $h = u + iv$ and let $\alpha = \int_a^b u(t)dt$, $\beta = \int_a^b v(t)dt$, then note that the left hand-side squared equals $\int_a^b (\alpha u(t) + \beta v(t))dt$. Now use the Cauchy-Schwarz Inequality.

2. Use the stereographic projection to show that the chordal distance between $z, w \in \mathbb{C}$ is given by the formula

$$\chi(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}.$$

Also, compute the chordal distance from a point $z \in \mathbb{C}$ and ∞.

3. Prove that f has a power series expansion about z_0 with radius of convergence $r > 0$ if and only if $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$ has a power series expansion about z_0, with the same radius of convergence. (How must you define $g(z_0)$, in terms of the coefficients of the series for f to make this a true statement?)

4. Prove that power series have the following Mean Value Property:

If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is convergent in $\{z : |z-z_0| < R\}$ $(R > 0)$, then for $0 < r < R$ we have

$$f(z_0) = \int_0^{2\pi} f(z_0 + re^{i\theta})\frac{d\theta}{2\pi}.$$

Hint: use uniform convergence.

5. Suppose f has a power series expansion at 0 which converges in all of $\mathbb{C}$. Suppose also that $\int_{\mathbb{C}} |f(x+iy)|dxdy < \infty$. Prove $f \equiv 0$. Hint: use the Mean Value Property.

6. Suppose $\sum_{j=0}^{\infty} |a_j|^2 < \infty$. Show $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $\{z : |z| < 1\}$. Hint: use the Cauchy-Schwarz Inequality.

Also use the Mean Value Property to compute:

$$\lim_{r \nearrow 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

7. Suppose f is analytic in a connected open set U such that for each $z \in U$, there exists an n (depending upon z) such that $f^{(n)}(z) = 0$. Prove f is a polynomial. Hint: use the Uniqueness Theorem for power series.

8. Let f be analytic in a region U containing the point $z = 0$. Suppose $|f(1/n)| < e^{-n}$ for $n \geq n_0$. Prove $f(z) \equiv 0$. Hint: use the local behavior.

9. Show that there is no continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(z)^2 = z$ for all $z \in \mathbb{C}$. Hint: consider $e^{-i\pi t} f(e^{i2\pi t})$ for $t \in \mathbb{R}$ and recall that continuous functions send connected sets into connected sets.

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$$(i)\sum_{n=1}^{\infty}\frac{z^n}{n} \qquad (ii)\sum_{n=1}^{\infty}\frac{z^n}{n^2} \qquad (iii)\sum_{n=1}^{\infty} nz^n \qquad (iv)\sum_{n=1}^{\infty} 2^{n^2} z^n \qquad (v)\sum_{n=1}^{\infty} 2^{-n^2} z^n$$

Verify the following facts about these examples. The radius of convergence of each of the first three series is $R = 1$. When $z = 1$, the first series is the harmonic series which diverges, and when $z = -1$ the first series is an alternating series whose terms decrease in absolute value and hence converges. The second series converges uniformly and absolutely on $\{|z| = 1\}$. The third series diverges at all points of $\{|z| = 1\}$. The fourth series has radius of convergence $R = 0$ and hence is not a convergent power series. The fifth example has radius of convergence $R = \infty$ and hence converges for all $z \in \mathbb{C}$.

What is the radius of convergence of the series $\sum a_n z^n$ where

$$a_n = \begin{cases} 3^{-n} & \text{if } n \text{ is even} \\ \\ 4^n & \text{if } n \text{ is odd.} \end{cases}$$

This is an example where ratios of successive terms in the series does not provide sufficient information to determine convergence.

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